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A model of p-branes with closed-constraint algebra

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Abstract. A new model of p-branes is introduced and studied. The general solution of this model is equivalent to a particle solution of the covariant p-brane problem which we have obtained earlier. The constraint algebra obtained here is closed in the sense that its structure functions are field independent.

The geometrical formulation of the mechanical motion of p-dimensional extended objects (so called p-branes) is based on the minimization of the world volumes which are swept from pieces of moving p-dimensional surfaces. This formulation is very simple but great difficulties arise when one tries to solve this problem even at the classical level. No general solutions were obtained except for the case $p=1$ (i.e. string). This is because in the higher p-branes there are essential nonlinearities. A related problem is that the Poisson brackets of the constraints are not linear functions of them, i.e. the structure 'constants' of the constraint algebra are field dependent.

In our opinion to overcome these difficulties we must define some particular model which must first preserve some of the properties of the covariant problem and, second, have a general solution which can be written down in a closed form. Moreover, this general solution must coincide with some particular solution of the covariant problem. Very convenient solutions for such a purpose are those obtained in a work of Stoyanov (1989) which were called characteristics. These characteristics are the generalization of the chiral bosonic string solutions depending on one of the cone variables only.

Let us recall how the above-mentioned solutions were obtained. First of all one introduces an appropriate gauge condition which fixes the determinant value of the space-like part of the induced world volume metric. Then the corresponding action can be written down in the following way:

$$A = T\int d^s\sigma[\sqrt{|\det(h_{\alpha\beta})|}+\lambda(\sqrt{|\det(h_{ij})|}-1)] \tag{1}$$

where $h_{\alpha\beta}=\partial_\alpha x^\mu\partial_\beta x_\mu$ is the induced metric on the world volume; $x_\mu(\sigma)$ are embeddings of the s-dimensional world volume into d-dimensional Minkowski space ($d>s$); the initial Greek indices $\alpha,\beta,\gamma,\ldots$ take the values from 0 to $s-1$ whereas the indices $\mu,\nu,\omega,\ldots$ are from 0 to $d-1$ (in the background Minkowski space); the indices $i,j,k,\ldots$ run from 1 to $s-1$; λ is a Lagrange multiplier and T is a constant with dimension of $(\text{mass})^s$.

As was proved by Stoyanov (1989), the minimization problem of the action (1) has an exact solution, and it was obtained there when the additional condition

$$\lambda=\frac{1}{2T}(\dot{x}_\mu(\delta^\mu_\nu-\partial_i x^\mu h^{ij}\partial_j x_\nu)\dot{x}^\nu)^{1/2} \tag{2}$$

took place and λ was an arbitrary function depending on σ^0 only. In the equality (2) h^{ij} is the inverse matrix of the space-like part of the induced metric i.e.

$$h^{ij}\partial_j x^\mu \partial_k x_\mu = \delta^i_k \qquad (i, j, k = 1, 2, \ldots, s-1) \tag{3}$$

and $\dot{x}_\mu = \partial_0 x_\mu$. In this case the new action

$$A' = T \int \sqrt{\dot{x}_\mu \Pi^\mu_\nu \dot{x}^\nu}\, \mathrm{d}^s\sigma \tag{4}$$

where $\Pi^\mu_\nu = \delta^\mu_\nu - \partial_i x_\nu h^{ij} \partial_j x^\mu$ is a projection matrix, is equivalent to the initial action A from (1). This means that from the two actions A, with condition (2), and A' follow the same equations of motion

$$\dot{p}_\mu - J^i \partial_i p_\mu = 0 \tag{5}$$

$$\partial_i J^i = 0 \tag{6}$$

and the same constraints

$$p^2 \equiv p_\mu p^\mu = T^2 \tag{7}$$

$$p_\mu \partial_i x^\mu = 0. \tag{8}$$

Here we have denoted

$$p_\mu = T \frac{\Pi^\nu_\mu \dot{x}_\nu}{\sqrt{x^\lambda \Pi^\omega_\lambda x_\omega}}. \tag{9}$$

Of course this statement is valid when the gauge condition

$$\sqrt{|\det(h_{ij})|} = 1 \tag{10}$$

is fulfilled.

A large set of solutions of the equations (5) and (6) was obtained by Stoyanov (1989) and they have the following form:

$$p_\mu = p_\mu(z^i) \tag{11}$$

$$x_\mu = G(\sigma^0) p_\mu(z^i) + t_\mu(z^i) \tag{12}$$

where

$$z^i = f^i_0(\sigma^0) + f^i(\sigma^0, \sigma^j). \tag{13}$$

Here $p_\mu(z^i)$ and $t_\mu(z^i)$ are arbitrary functions of the $s-1$ variables z^i; $f^i_0(\sigma^0)$ and $f^i(\sigma^0, \sigma^j) \equiv f^i(\sigma^0, \sigma^1, \ldots, \sigma^{s-1})$ are arbitrary functions, but satisfy the following condition:

$$|\det(\partial_i f^j)| = 1. \tag{14}$$

The functions $f^i_0(\sigma^0)$ and $f^i(\sigma^0, \sigma^j)$ are connected with the remaining gauge freedom.

From the equivalence between the two formulations it is easy to see that the solutions (11) and (12) when

$$G(\sigma^0) \equiv \text{constant}$$

(i.e. $\lambda=0$) are the particular ones of the covariant problem. Indeed (12) takes the new form:

$$x_\mu = x_\mu(z^i) \tag{15}$$

where $x_\mu(z^i)$ are a new set of arbitrary functions. Then (11) and (15) form the above-mentioned particular solution of the covariant problem. In the mathematical literature such solutions are known as characteristics (see e.g. Smirnov (1972)). For example if $s=1$, from equations (15), (13) and (14), we have one of the two chiral solutions

$$x_\mu = x_\mu(\sigma^0+\sigma^1) \qquad \text{or} \qquad x_\mu = x_\mu(\sigma^0-\sigma^1)$$

which are the characteristics for the two-dimensional d'Alembert equation. Moreover the gauge condition (10) is not of vital importance in this case and that is why we shall not consider it. The conditions (7) and (8) without the condition (10) already form a closed infinite Lie algebra.

In the present work we shall try to introduce, from previous considerations, an independent mechanical system which must be equivalent to the system with action from (4) at the level of the classical solutions (11) and (12) and constraints (7) and (8). This means that we must construct such a mechanical system with the same constraints for which the expressions (11) and (12) must form the general solution.

To construct our model we start from the Lagrangian formulation. Let us consider the following action:

$$S=\int [p^\mu \dot{x}_\mu - \lambda^0(p^2-T^2) - \lambda^i p^\mu \partial_i x_\mu]\,\mathrm{d}^s\sigma. \tag{16}$$

Here the quantities p_μ, x_μ, λ^0 and λ^i are independent fields defined on an s-dimensional manifold with coordinates σ^α. Note that the only difference between (16) and the usual p-brane action is that in the latter case the T^2 term in (16) is replaced by $\det|\partial_i x^\mu \partial_j x_\mu|$.

The action (16) has the same reparametrization invariance as (1), namely

$$\begin{aligned}
&\sigma^0 \to \xi^0 = \xi^0(\sigma^0)\\
&\sigma^i \to \xi^i = \xi^i(\sigma^0,\sigma^k) \qquad \text{with } \det\left|\frac{\partial \xi^i}{\partial \sigma^k}\right| = 1\\
&p_\mu(\sigma^\alpha) \to p_\mu(\xi^\alpha)\\
&x_\mu(\sigma^\alpha) \to x_\mu(\xi^\alpha)\\
&\lambda^0(\sigma^\alpha) \to \dot{\xi}^0(\sigma^0)\lambda^0(\xi^\alpha)\\
&\lambda^i(\sigma^\alpha) \to \dot{\xi}^0(\sigma^0)\frac{\partial \xi^i(\sigma^\beta)}{\partial \sigma^j}\lambda^j(\xi^\alpha) + \dot{\xi}^i(\xi^\alpha).
\end{aligned}$$

From the action (16) we have the following system of equations of motion:

$$p^2 = T^2 \tag{17}$$

$$p^\mu \partial_i x_\mu = 0 \tag{18}$$

$$\dot{p}_\mu - \partial_i \lambda^i p_\mu - \lambda^i \partial_i p_\mu = 0 \tag{19}$$

$$\dot{x}_\mu - \lambda^i \partial_i x_\mu - \lambda^0 p_\mu = 0. \tag{20}$$

The corresponding canonical momenta then have the form:

$$\pi_x^\mu \equiv \frac{\delta S}{\delta \dot{x}_\mu} = p^\mu \tag{21}$$

$$\pi_p^\mu \equiv \frac{\delta S}{\delta \dot{p}_\mu} = 0 \tag{22}$$

$$\pi_{\lambda_0} \equiv \frac{\delta S}{\delta \dot{\lambda}_0} = 0 \tag{23}$$

$$\pi_{\lambda^i} \equiv \frac{\delta S}{\delta \dot{\lambda}^i} = 0. \tag{24}$$

The last four equations are constraints. The first one means that the field p^μ must be identified with the canonical momentum of the field x^μ and the rest are trivial.

The first two equations of motion in our model coincide with the constraints (7) and (8). To prove that the system of (17) to (20) is equivalent to the system of (5) and (6) it is necessary to solve (19) and (20) for λ^0 and λ^i. One obtains the following expressions:

$$\lambda^0 = \frac{1}{2T}\sqrt{\dot{x}_\nu \Pi^\nu_\mu \dot{x}^\mu} \tag{25}$$

$$\lambda^i = \dot{x}_\mu h^{ij}\partial_j x^\mu \qquad \text{and} \qquad \partial_i \lambda^i = 0. \tag{26}$$

Finally one obtains the expression (9) for the momentum.

From (25) and (26) we can see that the additional condition (2) proposed for consideration by Stoyanov (1989) appears here automatically from the equations of motion. Moreover the fields λ^i coincide with the quantities denoted as J^i from the same paper.

Now we can write down the corresponding Hamilton function; in our case it has the following form:

$$H = \int [\lambda_0(p^2 - T^2) + \lambda^i p^\mu \partial_i x_\mu]\, d^{s-1}\sigma. \tag{27}$$

Considering the equations (17) and (18) as constraints we can construct their Poisson bracket with H. Introducing the following notation:

$$\psi \equiv p^2 - T^2 \qquad \text{and} \qquad \varphi_i \equiv p^\mu \partial_i x_\mu \tag{28}$$

we have

$$\{H, \psi\} = 2(\psi + T^2)\partial_i \lambda^i + \lambda^i \partial_i \psi \tag{29}$$

and

$$\{H, \varphi_i\} = (\partial_k \lambda^k \delta_i^j + \partial_i \lambda^j)\varphi_j + \lambda^j \partial_j \varphi_i + \lambda^0 \partial_i \psi + 2\partial_i \lambda^0(\psi + T^2) \tag{30}$$

where $\{.,.\}$ denotes the Poisson bracket.

We see that the following condition appears on the right-hand side of (30) as a secondary constraint:

$$\partial_i \lambda^0 = 0 \tag{31}$$

i.e. the expression (25) depends on σ^0 only. But this condition does not lead to an additional restriction because the solution (15) satisfies (31) identically.

We see that (17) and (18) are first class constraints according to the Dirac classification (Dirac 1964). As we have noted above, the corresponding algebra is closed. To construct the latter we introduce the generating functional of our constraints which has a similar form to the Hamilton function:

$$U(\mu^0, \mu^i) = \int (\mu^0 \psi + \mu^i \varphi_i)\, \mathrm{d}^{s-1}\sigma \tag{32}$$

where μ^0 and μ^i are arbitrary test functions. The quantity $U(\mu^0, \mu^i)$ coincides with H when μ^0 and μ^i satisfy conditions (26) and (31).

Remark. For our solutions (15) the integrals (32) do not depend on σ^0. With simple calculations one can obtain the following Poisson bracket

$$\begin{aligned} \{U(\mu^0, \mu^i), U(\nu^0, \nu^i)\} &= 2\int (\nu^0 \partial_i \mu^i - \mu^0 \partial_i \nu^i)(\psi + T^2)\, \mathrm{d}^{s-1}\sigma + \int (\nu^0 \mu^i - \mu^0 \nu^i)\partial_i \psi \, \mathrm{d}^{s-1}\sigma \\ &\quad + \int (\nu^j \partial_i \mu^i - \mu^j \partial_i \nu^i)\varphi_j\, \mathrm{d}^{s-1}\sigma + \int (\nu^i \mu^j - \nu^j \mu^i)\partial_j \varphi_i\, \mathrm{d}^{s-1}\sigma. \end{aligned} \tag{33}$$

Hitherto we have not mentioned the boundary conditions. Certainly all the integrals above must be taken in appropriate limits corresponding to the chosen boundary conditions. As a rule one applies the latter ones to specify the solutions of the equations of motion, then one expresses all mechanical quantities, including constraints, through specified solutions. In our case we shall act in another way; namely, different kinds of boundary conditions will be taken into account with the choice of an appropriate set of test functions for the functional (32).

The most simple example is one with periodic boundary conditions (in particular a torus compactified extended object). In this case the test functions will be a series of periodic exponents

$$\begin{aligned} \mu^0(\boldsymbol{n}, \boldsymbol{\sigma}) &= \mathrm{e}^{\mathrm{i}\boldsymbol{n}\boldsymbol{\sigma}} \\ \mu_j^k(\boldsymbol{n}, \boldsymbol{\sigma}) &= \delta_j^k\, \mathrm{e}^{\mathrm{i}\boldsymbol{n}\boldsymbol{\sigma}} \end{aligned} \tag{34}$$

where $\boldsymbol{n}$ denotes the $(s-1)$-dimensional vector with components $n_1, n_2, \ldots, n_{s-1}$ which are integer numbers; $\boldsymbol{\sigma}$ denotes the $(s-1)$-dimensional vector with components $\sigma_1, \sigma_2, \ldots, \sigma_{s-1}$ and

$$\boldsymbol{n}\boldsymbol{\sigma} = n_1\sigma_1 + n_2\sigma_2 + \ldots + n_{s-1}\sigma_{s-1}.$$

Using the series of functions (34) we can obtain the infinite-dimensional Lie algebra corresponding to the functional (32). Let us define the following notation:

$$U(0, \mu_{i,\boldsymbol{n}}) \equiv \varphi_i^{\boldsymbol{n}} = \int \mathrm{e}^{\mathrm{i}\boldsymbol{n}\boldsymbol{\sigma}} \varphi_i(\sigma)\, \mathrm{d}^{s-1}\sigma \tag{35}$$

$$U(\mu_{\boldsymbol{n}}^0, 0) \equiv \psi^{\boldsymbol{n}} = \int \mathrm{e}^{\mathrm{i}\boldsymbol{n}\boldsymbol{\sigma}} \psi(\sigma)\, \mathrm{d}^{s-1}\sigma. \tag{36}$$

Then from equations (33-36) it is easy to obtain the following Poisson brackets for the quantities φ_i^n and ψ^n:

$$\{\psi^n, \psi^m\} = 0 \tag{37}$$

$$\{\varphi_i^n, \varphi_j^m\} = \mathrm{i} \cdot n_j \varphi_i^{n+m} - \mathrm{i} \cdot m_i \varphi_j^{n+m} \tag{38}$$

$$\{\psi^n, \varphi_i^m\} = \mathrm{i}(n_i - m_i)\psi^{n+m} - 8\pi^2 \mathrm{i} T^2 m_i \delta_{n+m,0} \tag{39}$$

where

$$\delta_{n+m} = \delta_{n_1+m_1,0}\delta_{n_2+m_2,0} \ldots \delta_{n_{s-1}+m_{s-1},0}.$$

The obtained algebra contains some well known algebras (Floratus *et al* 1988, Hoppe 1988, Bars *et al* 1988) as subalgebras. First of all when $i = j$, from (38), we have

$$\{\varphi_i^n, \varphi_i^m\} = \mathrm{i}(n_i - m_i)\varphi_i^{n+m}. \tag{40}$$

This is a trivial generalization of the Witt algebra in $(s-1)$-dimensional space. A more interesting subalgebra arises after we introduce the notation:

$$J_n^{ij} = n_k A_{kl}^{ij} \varphi_l^n \tag{41}$$

where the matrices

$$A_{kl}^{ij} = \delta_k^i \delta_l^j - \delta_l^i \delta_k^j \tag{42}$$

are the generators of the group $\mathrm{O}(s-1)$. Then from (38) and (41) we can obtain (see also Hoppe (1988))

$$\{J_n^{ij}, J_m^{ij}\} = (nA^{ij}m)J_{n+m}^{ij}. \tag{43}$$

Here

$$(nA^{ij}m) = \sum_{kl} n_k A_{kl}^{ij} m_l.$$

In particular in the case of a membrane (i.e. $s = 2$) there is only one matrix (42) which coincides with

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the subalgebra (43) coincides with the algebra studied by Floratos *et al* (1988), Hoppe (1988), Bars *et al* (1988).

Finally we can obtain the most general form of the central charge:

$$C(J_n^{ij}, J_m^{ij}) = a^i(\boldsymbol{n}+\boldsymbol{m})n_i\delta_{n_i+m_i,0} + a^j(\boldsymbol{n}+\boldsymbol{m})n_j\delta_{n_j+m_j,0} + (\boldsymbol{b}^{ij} \cdot \boldsymbol{n})\delta_{\boldsymbol{n}+\boldsymbol{m},0} \tag{44}$$

where $a^i(\boldsymbol{n}+\boldsymbol{m})$ are arbitrary functions and $\boldsymbol{b}^{ij}$ is an arbitrary, constant vector. Note that in the first two terms of central charge (44) there is no summation over the indices i and j. The first and second term on the RHS of (44) can be removed by a suitable redefinition of the generators but we keep them since they might be of physical interest, as we can see from Virassoro algebra case.

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References

Bars I, Pope C N and Sezgin E Central extensions of membrane symmetry algebras 1988 *Preprint* USC-88/HEP04

Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshive University Press)

Floratus E G and Iliopoulos J 1988 *Phys. Lett.* B **201** 237

Hoppe J 1988 *Phys. Lett.* B **215** 706

Smirnov V I 1972 *A Course in Higher Mathematics* vol IV (Moscow: Nauka)

Stoyanov D T 1989 *Mod. Phys. Lett.* A **4** 1287